MPI-PhT/95-109
November 1995

q-deformed Phase Space and its Lattice Structure

M. Fichtmüller[1], A. Lorek*[1] and J. Wess[1,2]

[1] Max-Planck-Institut für Physik
Werner-Heisenberg-Institut
Föhringer Ring 6 , D - 80805 München, Germany
Tel. (89) 32354-0, Fax (89) 3226704

[2] Sektion Physik, Universität München
Theresienstr. 37, D - 80333 München, Germany

Abstract

A q-deformed two-dimensional phase space is studied as a model for a non-commutative phase space. A lattice structure arises that can be interpreted as a spontaneous breaking of a continuous symmetry. The eigenfunctions of a Hamiltonian that lives on such a lattice are derived as wavefunctions in ordinary x-space.

*Supported by the German-Israeli Foundation (G.I.F.)

Introduction

Quantum groups are a generalization of the concept of symmetries. They act on noncommutative spaces that inherit a well-defined mathematical structure from the quantum group symmetries.
Starting from such a noncommutative space as configuration space we generalize it to a phase space where noncommutativity is already intrinsic for a quantum mechanical system. The definition of this noncommutative phase space is derived from the noncommutative differential structure on the configuration space.
The thus obtained noncommutative phase space is a q-deformation of the quantum mechanical phase space and we can apply all the machinery as we have learned it from quantum mechanics. We shall associate essentially selfajoint operators in a Hilbert space with observables and define the time development by a Schroedinger equation.

As an interesting result we shall see that the observables that are associated with position and momentum in a natural way have a discrete spectrum. The q-deformed phase space puts physics on a q-lattice. This lattice can be imbedded in an ordinary quantum mechanical phase space and q-deformation has many similarities to a spontaneous breaking of a symmetry.
The operators exhibiting the discrete spectrum will transform nonlinearly under a continuous translation. For certain Hamiltonians the time development of the system takes place entirely on the lattice. This shows that dynamics can deform continuous space to a lattice structure.

In chapter 1 we define the concept of a q-deformed two-dimensional phase space. In chapter 2 we study the Hilbert space representations of the q-deformed phase space operators. This shows their discrete spectrum. We also show that these operators can be represented in the usual Hilbert space of nondeformed quantum mechanics [1]. In chapter 3 we show that these representations can be reduced to the representations mentioned before.
In chapter 4 we show that translational invariance can be defined in the thus obtained subspaces of the ordinary Hilbert space and that the q-deformed variables transform nonlinearly.
In chapter 5 we finally study a simple quantum mechanical system formulated in terms of q-deformed phase space variables and we find the eigenfunctions of the corresponding Hamiltonian as wavefunctions in the usual quantum mechanical formulation.

1 The Algebra

The simplest q-deformed differential calculus in one dimension is based on the following Leibniz rule [2]:

$$\partial_{\rm x}\,{\rm x} = 1 + q\,{\rm x}\,\partial_{\rm x} \tag{1.1}$$

This could be the starting point for a q-deformed Heisenberg algebra. However, if x is assumed to be a hermitean operator in a Hilbert space the usual quantization rule $p \to -i\,\partial_{\rm x}$ does not yield a hermitean momentum operator. This can be seen by comparing (1.1) with its conjugate relation:

$${\rm x}\,\overline{\partial}_{\rm x} = 1 + q\,\overline{\partial}_{\rm x}\,{\rm x} \quad , \qquad \overline{q} = q \tag{1.2}$$

Nevertheless we are going to define a conjugation operation on the algebra x, $\partial_{\rm x}$ which is consistent with (1.2). This is done with the help of a scaling operator:

$$\Lambda = 1 + (q-1)\,{\rm x}\,\partial_{\rm x} \tag{1.3}$$

From (1.1) follows

$$\Lambda\,{\rm x} = q\,{\rm x}\,\Lambda \quad , \qquad \Lambda\,\partial_{\rm x} = q^{-1}\,\partial_{\rm x}\,\Lambda \tag{1.4}$$

The occurence of such a scaling operator is typical for a noncommutative differential calculus derived from quantum group symmetries. For $q = 1$ we would find $\Lambda = 1$, in the undeformed case the scaling operator Λ is not at our disposal.

If we define conjugation by

$$\overline{\partial}_{\rm x} = -q^{-1}\Lambda^{-1}\partial_{\rm x} \quad , \qquad \overline{\rm x} = {\rm x} \tag{1.5}$$

then the equations (1.1) and (1.2) are consistent. Conjugation, being an involution, tells us that

$$\partial_{\rm x} = -q^{-1}\overline{\partial}_{\rm x}\,\overline{\Lambda}^{-1} \tag{1.6}$$

This is only possible if

$$\overline{\Lambda} = q^{-1}\,\Lambda^{-1} \tag{1.7}$$

which can be verified by a direct calculation.

The obvious choice for a hermitean momentum operator is

$$P = -\frac{i}{2}\,(\partial_{\rm x} - \overline{\partial}_{\rm x}) \tag{1.8}$$

where $\overline{\partial}_{\rm x}$ has to be taken from (1.5). With equation (1.3) this is an expression in terms of x and $\partial_{\rm x}$.

Commuting P through x yields:

$$\begin{aligned} P\,{\rm x} &= -\frac{i}{2}\,(1+\frac{1}{q}) - \frac{i}{2}\,{\rm x}\,(q\,\partial_{\rm x} - \frac{1}{q}\,\overline{\partial}_{\rm x}) \\ &= -\frac{i}{2}\,(1+\frac{1}{q}) + q\,{\rm x}\,P - \frac{i}{2}\,(q-\frac{1}{q})\,{\rm x}\,\overline{\partial}_{\rm x} \end{aligned} \tag{1.9}$$

We find

$$P\,\mathrm{x} - q\,\mathrm{x}\,P = -\frac{i}{2}\,(1+\frac{1}{q})\,\left[1+(q-1)\,\mathrm{x}\,\overline{\partial}_{\mathrm{x}}\right] \tag{1.10}$$

For hermitean P and x the right hand side of the q-deformed commutator had to be an operator. Any such operator might be expressed in terms of the product of a unitary and a hermitean operator. Using (1.2) and (1.3) we see that the right hand side of (1.10) can be expressed in terms of $\overline{\Lambda}$.

With a simple redefinition

$$U = q^{\frac{1}{2}}\,\overline{\Lambda} \quad , \qquad X = \frac{1+q}{2q}\,\mathrm{x} \tag{1.11}$$

we arrive at a q-deformed Heisenberg relation:

$$\begin{gathered} q^{\frac{1}{2}}\,X\,P - q^{-\frac{1}{2}}\,P\,X = i\,U \\ U\,X = q^{-1}\,X\,U \quad , \qquad U\,P = q\,P\,U \end{gathered} \tag{1.12}$$

with conjugation properties

$$\overline{P} = P \;\; , \quad \overline{X} = X \;\; , \quad \overline{U} = U^{-1} \tag{1.13}$$

This will be the starting point of our investigations.

A q-deformed quantization forced us to introduce an additional unitary operator U, very much in the same way as ordinary quantization forces us to a purely imaginary right hand side of the commutator.

2 The Representations

Representations of the algebra (1.12) have been constructed in references [3], [4], [5]. We are shortly listing the results.

As we are interested in representations where P and X are represented by essentially selfajoint operators (so that they can be diagonalized) we may start as well with a representation where P is diagonal. We note that a rescaling of P and X by a real parameter s ($P \to sP$, $X \to s^{-1}X$) does not change the algebra nor the conjugation properties. Therefore we can always scale a nonvanishing eigenvalue to $+1$. It follows from the commutation property of P and U, that such a representation will have all the eigenvalues q^n ($n \in \mathbb{Z}$). The corresponding eigenstates form a basis of a representation space for the full algebra. Analogously we could have obtained a representation with eigenvalues $-q^n$.

These representations we denote by $\hat{P}$, $\hat{X}$, $\hat{U}$, the corresponding eigenstates by

$|n, \sigma>$, $\sigma = +, -$ respectively. It is easy to check that these representations are of the following form:

$$\begin{aligned}
\hat{P}\,|n,\sigma> &= \sigma\, q^n\, |n,\sigma> \\
\hat{X}\,|n,\sigma> &= i\sigma \frac{q^{-n}}{q-q^{-1}} \left(q^{\frac{1}{2}} |n-1,\sigma> - q^{-\frac{1}{2}} |n+1,\sigma> \right) \\
\hat{U}\,|n,\sigma> &= |n-1,\sigma> \qquad\qquad (2.1) \\
< n', \sigma' \,|\, n,\sigma> &= \delta_{nn'}\,\delta_{\sigma\sigma'}
\end{aligned}$$

For each choice of σ, this forms a representation. However for a fixed value of σ, $\hat{X}$ is not essentially selfadjoint. There is a one parameter family of selfadjoint extensions - none of which satisfies the algebra. It is, however, possible to find a selfadjoint extension that satisfies the algebra in a representation where σ takes both values. This will be our representation space in what follows. When diagonalized, $\hat{X}$ will have the eigenvalues $\mp\lambda^{-1} q^{-\frac{1}{2}} q^{\nu}$ $(\nu \in \mathbb{Z})$. With the help of q-deformed cosine and sine functions, the change of basis can be achieved.

The elements of the q-deformed algebra (1.12) can also be expressed in terms of the operators of an undeformed algebra, which we denote by p, x; they satisfy $[x, p] = i$. (This operator x should not be confused with the element x in chapter 1.)
The relevant relations are:

$$\begin{aligned}
P &= p \\
X &= \frac{[z+\frac{1}{2}]}{z+\frac{1}{2}}\, x \qquad\qquad (2.2) \\
U &= q^z
\end{aligned}$$

where

$$z = -\frac{i}{2}\,(xp + px) \qquad\qquad (2.3)$$

$$[A] = \frac{q^A - q^{-A}}{q - q^{-1}} \qquad \text{for any A}$$

The algebraic relations (1.12) follow from the canonical commutation properties of p, x. We outline the proof [6]. We start from

$$x\,z = (z+1)\,x \quad , \quad p\,z = (z-1)\,p \qquad\qquad (2.4)$$

so that

$$x\,f(z) = f(z+1)\,x \quad , \quad p\,f(z) = f(z-1)\,p \qquad\qquad (2.5)$$

for functions $f(z)$. We also use the identity $z = -ipx + \frac{1}{2} = -ixp - \frac{1}{2}$. Inserting (2.2) into (1.12) yields

$$q^{\frac{1}{2}} \frac{[z+\frac{1}{2}]}{z+\frac{1}{2}}\, xp - q^{-\frac{1}{2}} \frac{[z-\frac{1}{2}]}{z-\frac{1}{2}}\, px = iq^{\frac{1}{2}}\,[z+\frac{1}{2}] - iq^{-\frac{1}{2}}\,[z-\frac{1}{2}] \;=\; iq^z \qquad\qquad (2.6)$$

The conjugation properties (1.13) follow from the hermiticity of x and p.

Algebraically the form of the relations (2.2) can be changed by canonical transformations on x and p. An interesting class of such canonical transformations is:

$$\tilde{p} = f(z)\,p \;\; , \quad \tilde{x} = x\,f^{-1}(z) \tag{2.7}$$

The hermiticity of $\tilde{p}$ and $\tilde{x}$ demands:

$$\overline{f}(\overline{z}) = f(z+1) \tag{2.8}$$

This condition is e.g. satisfied for

$$f^{-1}(z) = \frac{[z-\frac{1}{2}]}{z-\frac{1}{2}} \tag{2.9}$$

With this choice for a canonical transformation the relations (2.2) become:

$$\begin{aligned} P &= \frac{[\tilde{z}-\frac{1}{2}]}{\tilde{z}-\frac{1}{2}}\,\tilde{p} \\ X &= \tilde{x} \\ U &= q^{\tilde{z}} \end{aligned} \tag{2.10}$$

The standard Hilbert space representation of p, x leads to a representation of P, X and U via the relation (2.2) or (2.10). How this representation is related to the representations (2.1) will be discussed in the next chapter.
In this context we will encounter rescaled eigenvalues of P, $\sigma s q^n$ $(1{\le}s{<}q)$. To distinguish such representations we introduce an operator S which commutes with P, X and U, which is hermitean and has a spectrum ranging from 1 to q.

$$\begin{aligned} &S^{+} = S \;\; , \;\; [S,P] = [S,X] = [S,U] = 0 \\ &S\,|s> = \; s\,|s> \qquad 1 \le s < q \\ &< s'\,|\,s> = \; \delta(s'-s) \end{aligned} \tag{2.11}$$

With $\hat{X}$, $\hat{P}$, $\hat{U}$ we will find another representation of (1.12):

$$\begin{aligned} P &= S\,\hat{P} \\ X &= S^{-1}\,\hat{X} \\ U &= \hat{U} \end{aligned} \tag{2.12}$$

The eigenstates of P are $|n,\sigma>|s>$ with the eigenvalue $\sigma s q^n$.

3 Reduction of Representations

Relations (2.2) allow to represent P, X, U in the Hilbert space of ordinary quantum mechanics where we choose a momentum representation

$$\begin{aligned} p\,|p_0> &= p_0\,|p_0> \\ <p_0'\,|\,p_0> &= \delta(p_0'-p_0) \end{aligned} \tag{3.1}$$

Equation (2.1) suggests to change the above basis to the new basis:

$$|n,\sigma>|s> = \int dp_0\, q^{\frac{n}{2}}\,\delta(p_0-\sigma s q^n)\;|p_0> \tag{3.2}$$

It is easy to check that with the normalization (3.1) the states $|n,\sigma>$ and $|s>$ are normalized as in (2.1) and (2.11). The inverse transformation is:

$$|p_0> = \int_1^q ds \sum_{n=-\infty}^{\infty}\sum_{\sigma=+,-} q^{\frac{n}{2}}\,\delta(p_0-\sigma s q^n)\;|n,\sigma>|s> \tag{3.3}$$

We now show that this change of basis reduces the representation of P, X, U in terms of x and p to representations as we encountered them in (2.1).
For the operator P, this is trivial:

$$P\,|n,\sigma>|s> = \;\sigma s q^n\,|n,\sigma>|s> \tag{3.4}$$

For the operators U and X it requires a short calculation. We know that x is represented by $i\frac{\partial}{\partial p}$ when acting on a wave function. Therefore:

$$U\,f(p) = q^{\frac{1}{2}} q^{p\frac{\partial}{\partial p}}\;f(p) = q^{\frac{1}{2}} f(qp) \tag{3.5}$$

This has as an immediate consequence that

$$U\,|n,\sigma>|s> = \;|n-1,\sigma>|s> \tag{3.6}$$

The operator X of equation (2.2) can be rewritten as follows:

$$X = \frac{i}{q-q^{-1}}\,\left(q^{\frac{1}{2}}\,U - q^{-\frac{1}{2}}\,U^{-1}\right)\;P^{-1} \tag{3.7}$$

This again implies (2.1) directly. The states $|n,\sigma>|s>$ carry a representation of P, X and U for a fixed value of s.

Relations (2.2) can be inverted formally:

$$\begin{aligned} p &= P \\ z &= h^{-1}\ln U \\ x &= \frac{i}{h}\ln\left(q^{\frac{1}{2}}U\right)\;P^{-1} \qquad , \qquad q = e^h \end{aligned} \tag{3.8}$$

Hermiticity properties of x and z may be used to make $\ln U$ less ambiguous.
Of course we cannot succeed in constructing selfadjoint representations of the canonical operators x and p in the representation space of (2.1). This manifests itself when we apply $\ln U$ to any of the states $|n, \sigma>$, we get an expansion of the logarithm around zero or infinity.

There are states, not normalizable, that are eigenstates of the operator U:

$$\begin{aligned} |\phi_0> = & \frac{1}{\sqrt{2\pi}} \int_1^q ds \sum_{n,\sigma} q^{\frac{n}{2}} \, |n, \sigma> |s> = \frac{1}{\sqrt{2\pi}} \int dp_0 \, |p_0> \\ U|\phi_0> = & \; q^{\frac{1}{2}} \, |\phi_0> \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} |\phi_m> = & \; P^m \, |\phi_0> \\ U|\phi_m> = & \; q^{m+\frac{1}{2}} \, |\phi_m> \end{aligned} \tag{3.10}$$

For these states, in a formal way, we find:

$$x \, |\phi_m> = \; im \, |\phi_{m-1}> \qquad\qquad m \geq 0 \tag{3.11}$$

Eigenstates of x can be obtained from the states $|\phi_m>$:

$$\begin{aligned} |x_0> = & \sum_{m=0}^{\infty} \frac{(-i)^m \, x_0^m}{m!} \, |\phi_m> \\ x \, |x_0> = & \; x_0 \, |x_0> \end{aligned} \tag{3.12}$$

After these heuristic arguments we show that the states $|x_0>$ are well defined. For this purpose it is convenient to use (3.3) and write $|x_0>$ in the following form:

$$\begin{aligned} |x_0> \;\; = & \;\; \frac{1}{\sqrt{2\pi}} \sum_{m=0}^{\infty} \frac{(-i)^m \, x_0^m}{m!} \, P^m \int dp_0 \int_1^q ds \sum_{n,\sigma} \delta(p_0 - \sigma s q^n) \; q^{\frac{n}{2}} \, |n, \sigma> |s> \\ = & \;\; \frac{1}{\sqrt{2\pi}} \int dp_0 \int_1^q ds \sum_{n,\sigma} e^{-ix_0 \, p_0} \, \delta(p_0 - \sigma s q^n) \; q^{\frac{n}{2}} \, |n, \sigma> |s> \end{aligned} \tag{3.13}$$

This is nothing but the Fourier transform of the states $|p_0>$. The relation

$$< x_0' \, | \, x_0> = \; \delta(x_0' - x_0) \tag{3.14}$$

follows after a short calculation from the normalization of the states $|n, \sigma> |s>$.

We have obtained a representation of x that is essentially selfajoint from a reducible representation of X and P.

4 Translational Invariance

The unitary operator that represents translation is e^{ipa}. From (3.8) follows that the representation space of P, X and U is left invariant under such a translation. This shows that there is a map of the operators P, X and U into operators P', X', U' that corresponds to a translation of the system:

$$\begin{aligned} P' &= e^{iap}\,P\,e^{-iap} = P \\ U' &= e^{iap}\,U\,e^{-iap} = U\,e^{iaq^{-1}(1-q)P} \\ X' &= e^{iap}\,X\,e^{-iap} = \frac{i}{\lambda}\left(q^{\frac{1}{2}}\,U\,e^{iaq^{-1}(1-q)P} - q^{-\frac{1}{2}}\,U^{-1}\,e^{ia(q-1)P}\right)\,P^{-1} \end{aligned} \tag{4.1}$$

This exemplifies an idea of Snyder who implemented Lorentz invariance in quantized spaces [7]. Formula (4.1) resembles formulas for nonlinear transformation laws:

$$\delta X = X' - X = q^{-\frac{1}{2}}\,(q+1)^{-1}\left(U + U^{-1}\right)\,a \tag{4.2}$$

For nonlinear transformation laws a shift in the origin is a characteristic feature. Here X is shifted by an operator.

Let us demonstrate the action of the translation on a state $|x_0>$ (3.13) by acting on the Hilbert space of (2.1):

$$\begin{aligned} e^{iap}\,|x_0> &= \frac{1}{\sqrt{2\pi}}\int dp_0 \int_1^q ds \sum_{n,\sigma} e^{-ix_0\,p_0}\,\delta(p_0-\sigma s q^n)\;q^{\frac{n}{2}}\,e^{iaP}\,|n,\sigma>|s> \\ &= \frac{1}{\sqrt{2\pi}}\int dp_0 \int_1^q ds \sum_{n,\sigma} e^{-ix_0\,p_0}\,\delta(p_0-\sigma s q^n)\;q^{\frac{n}{2}}\,e^{ia\sigma s q^n}\,|n,\sigma>|s> \\ &= \frac{1}{\sqrt{2\pi}}\int dp_0 \int_1^q ds \sum_{n,\sigma} e^{-i(x_0-a)\,p_0}\,\delta(p_0-\sigma s q^n)\;q^{\frac{n}{2}}\,|n,\sigma>|s> \\ &= |x_0-a> \end{aligned} \tag{4.3}$$

This clearly shows that the translation acts only on the lattice part of the decomposition (2.12) and not on the operator S - again a situation we are used to from realizing spontaneous symmetry breaking by nonlinearly transforming fields.

5 q-deformed Dynamics

The simplest q-deformed Hamiltonian is

$$H = \frac{1}{2}\,P^2 \tag{5.1}$$

We know that this Hamiltonian has eigenvalues $\frac{1}{2}s^2q^{2n}$. If we associate canonical variables as in (2.10) (in the following we drop the tilde on x and p) this describes an interacting system:

$$H = \frac{2}{\lambda^2}\; p\; \frac{q+q^{-1}-2\cos((xp+px)h)}{1+(xp+px)^2}\; p \tag{5.2}$$

We construct the eigenfunctions of this Hamiltonian in the x-representation. Through the identification (2.10) it is easy to find the eigenfunctions of the operator X

$$|\nu,\tau>|s>=\int dx_0\; \lambda^{-\frac{1}{2}}\, q^{\frac{\nu}{2}-\frac{1}{4}}\, \delta(s\, x_0 - \lambda^{-1}\tau q^{\nu-\frac{1}{2}})\; |x_0> \tag{5.3}$$

The normalization has been chosen such that

$$\begin{aligned} < x_0' \,|\, x_0 > &= \delta(x_0'-x_0) \\ < \nu',\tau' \,|\, \nu,\tau > &= \delta_{\nu'\nu}\, \delta_{\tau'\tau} \\ < s' \,|\, s > &= \delta(s'-s) \end{aligned} \tag{5.4}$$

and we have eigenstates of the operator X with eigenvalues $\tau q^{-\frac{1}{2}}\lambda^{-1}\, s^{-1}\, q^{\nu}$ as we expect them from the representation theory of (1.12).
In reference [4] we have learned how to transform such a basis into a basis of momentum eigenstates. Such a transformation will diagonalize the Hamiltonian. We choose the normalization of the X eigenstates consistent with the representation (2.10) of U.
The momentum eigenstates are:

$$\begin{aligned} |2m,\sigma> \;=\;& \frac{1}{2}\, N_q \sum_\nu q^{\nu+m} \Big[\cos_q(q^{(\nu+m)})\,(|2\nu,\tau{=}{+}> + \,|2\nu,\tau{=}{-}>) \\ & \qquad - i\sigma\; \sin_q(q^{2(\nu+m)})\,(|2\nu{+}1,\tau{=}{+}> - \,|2\nu{+}1,\tau{=}{-}>)\Big] \\ |2m{+}1,\sigma> \;=\;& \frac{1}{2}\, N_q \sum_\nu q^{\nu+m} \Big[\cos_q(q^{(\nu+m)})\,(|2\nu{-}1,\tau{=}{+}> + \,|2\nu{-}1,\tau{=}{-}>) \\ & \qquad - i\sigma\; \sin_q(q^{2(\nu+m)})\,(|2\nu,\tau{=}{+}> - \,|2\nu,\tau{=}{-}>)\Big] \end{aligned} \tag{5.5}$$

The functions $\cos_q$ and $\sin_q$ were defined by Koornwinder and Swarttouw [8] with the following change of notation: $\cos_q(z) = \cos(z;q^{-4})$, $\sin_q(z) = \sin(z;q^{-4})$.
Of importance is the following relation

$$\begin{aligned} \tfrac{1}{z}\,[\cos_q(z) - \cos_q(q^{-2}z)] &= -q^{-2}\, \sin_q(q^{-2}z) \\ \tfrac{1}{z}\,[\sin_q(z) - \sin_q(q^{-2}z)] &= \cos_q(q^{-2}z) \end{aligned} \tag{5.6}$$

When we insert the wavefunction (5.3) into (5.5) we get the wavefunctions for the momentum eigenstates.

For the eigenstate of the Hamiltonian we consider the state

$$\begin{aligned} &\frac{1}{\sqrt{2}}\,(\,|2m,\sigma{=}+> |s> + |2m,\sigma{=}-> |s>\,) \qquad (5.7) \\ &= \frac{1}{\sqrt{2}} N_q \sum_{\nu,\tau} q^{\nu+m} \cos_q(q^{2(\nu+m)}) \int dx_0\, \delta(sx_0 - \tau\lambda^{-1} q^{2\nu-\frac{1}{2}})\lambda^{-\frac{1}{2}}\, q^{\nu-\frac{1}{4}}\, |x_0> \end{aligned}$$

The corresponding eigenfunction of the Hamiltonian (5.2) to the eigenvalue $\frac{1}{2}s^2q^{4m}$ is:

$$\Psi_{2m,s}(x) = \sqrt{\frac{\lambda}{2}}\, N_q \sum_{\nu,\tau} \tau sx\, q^{m+\frac{1}{4}}\ \cos_q(q^{2m+\frac{1}{2}}\lambda\tau sx)\, \delta(sx - \tau\lambda^{-1}q^{2\nu-\frac{1}{2}}) \qquad (5.8)$$

Forgetting the way we obtained this wavefunction we can verify that it is indeed an eigenfunction of the Hamiltonian (5.2) by making use of (5.6). The Hamiltonian when applied to a function acts in the following way

$$Hf(x) = -\frac{1}{2\lambda^2}\frac{1}{x^2}\left(q\, f(\frac{1}{q^2}x) - (q+\frac{1}{q})\, f(x) + \frac{1}{q}\, f(q^2x)\right) \qquad (5.9)$$

A short calculation verifies:

$$H\ \Psi_{2m,s} = \frac{s^2}{2}\, q^{4m}\ \Psi_{2m,s} \qquad (5.10)$$

The eigenfunctions are normalized to a δ-function in the s-space. In a similar way we can treat all the eigenfunctions of H.

It is interesting to note that the Hamiltonian (5.2) forces the system to live on a lattice. If there is an interaction which is a polynomial in P and X the system will never leave this lattice because the eigenvalue s of S will not change.

References

[1] A. Lorek, J. Wess:
Dynamical Symmetries in q-deformed Quantum Mechanics,
Z. Phys. C 67 (1995) 671

[2] J. Wess, B. Zumino:
Covariant Differential Calculus on the Quantum Hyperplane,
Nucl. Phys. B (Proc. Suppl.) 18B (1990) 302

[3] J. Schwenk, J. Wess:
A q-deformed Quantum Mechanical Toy Model,
Phys. Lett. B 291 (1992) 273

[4] A. Hebecker, S. Schreckenberg, J. Schwenk, W. Weich, J. Wess:
Representations of a q-deformed Heisenberg Algebra,
Z. Phys. C 64 (1994) 355

[5] J. Schwenk:
q-deformed Fourier Theory,
preprint MPI-PhT/94-36, hep-th/9406168

[6] T. Curtright, C. Zachos:
Deforming Maps for Quantum Algebras,
Phys. Lett. B 243 (1990) 237

[7] H. S. Snyder:
Quantized Space-Time,
Phys. Rev. 71 (1947) 38

[8] T. H. Koornwinder, R. F. Swarttouw:
On q-analogues of the Fourier and Hankel Transforms,
Trans. AMS 333 (1992) 445